A Finitely Presented Orderable Group with Insoluble Word Problem.

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October 5, 2010

Dedicated to James Wiegold, in memoriam.

Abstract

We construct a finitely presented (two-sided) totally orderable group with insoluble word problem.

1 Introduction

Let G be a group with a total order $<$. Then G is said to be a *right-ordered group* (with respect to $<$) if $f < g$ implies $fx < gx$ for all $x \in G$. We can similarly define a left-ordered group. A group with a total order that is respected by both right and left multiplication is called a *two-sided ordered group*, or an *ordered group* or o-*group*, for short. A group that can made into a right-ordered group with respect to some total order is called a *right-orderable* group, and a group that can be made into an o-group with respect to some total order is called a (*two-sided*) *orderable* group. In [2] and [3], we constructed finitely presented right-orderable groups with insoluble word problem. These examples are not two-sided orderable. Here we prove

2010 AMS Classification: 20F10, 06F15, 20F60.

Keywords: Finitely presented group, HNN-extension, right-orderable group, (two-sided) orderable group, insoluble word problem, basic commutator.

This research was supported by a grant from the Royal Society for which we are most grateful. It was done while the first author was a Distinguished Academic Visitor at Queens' College, Cambridge; he wishes to thank the College and DPMMS for their warm hospitality.

Theorem A *There is a finitely presented (two-sided) orderable group with insoluble word problem.*

Our proof involves taking a finitely presented *right*-orderable group F/N with insoluble word problem and using it to obtain a finitely presented (two-sided) orderable group with insoluble word problem. We do this as follows. Let F be the free group on m generators and $u_1, \ldots, u_n$ be the generators of N (as a *normal* subgroup of F). Let G_0 be a semidirect product of F by a free group on $2m$ generators that normalises N. Take an HNN-extension G_1 of G_0 with stable letter t that fixes each element of N. Let T_0 be the normal closure of $\langle t \rangle$ by G_0 (equivalently, in G_1). We use the right order on F/N to give a right order $\prec$ on G_0 and thence an order on the generators of the free group T_0 by $t^{f_1} < t^{f_2}$ if and only if $f_1 \prec f_2$. We will give an ordering of basic commutators in a free group and derive a G_1-invariant order on T_0. We embed T_0 into its topological completion T_0^*, choose an appropriate subgroup $T < T_0^*$, and form a semidirect product of T by a direct product of G_0 and a free group $F(\bar{y})$ on $2m$ generators. This is our (two-sided) orderable group G. We derive the infinite set of relations $t^{-1}ut = u$ for all $u \in N$ from a *finite* set of defining relations of G which include $t^{-1}u_i t = u_i$ $(i = 1, \ldots, n)$. In our construction, t commutes (in G) with $x \in F$ if and only if $x \in N$. It follows that G has insoluble word problem.

2 Preliminaries (Groups)

We will write $\ell(w)$ for the *length* of a reduced word w in a free group.

For general information about HNN-extensions, see [9], Chapter IV, Section 2. We summarise what we require.

Let G be a group with isomorphic subgroups A and B, say $\varphi : A \cong B$. The *HNN-extension* of G relative to A, B, φ is the group

$$G^{\#} = \langle G, t;\, t^{-1}at = a^{\varphi}\ (a \in A)\rangle.$$

Definition ([9], page 181) A sequence $g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n$ $(n \geq 0, \varepsilon_i \in \{\pm 1\},\ i = 1, \ldots, n)$ is said to be *reduced* if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_i, t^{-1} with $g_i \in B$.

Britton's Lemma ([9], page 181) *If the sequence $g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n$ is reduced and $n \geq 1$, then $g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n \neq 1$ in $G^{\#}$.*

If C is any subgroup of a group G, we always let 1 be the right or left coset representative of C.

Definition. ([9], page 181) A *normal form* is a sequence

$$g_0,\, t^{\varepsilon_1},\, g_1,\, \ldots,\, t^{\varepsilon_n},\, g_n \quad (n \geq 0), \qquad \text{where}$$

(i) g_0 is an arbitrary element of G,
(ii) if $\varepsilon_i = -1$, then g_i is a representative of a right coset of A in G,
(iii) if $\varepsilon_i = +1$, then g_i is a representative of a right coset of B in G, and
(iv) there is no consecutive subsequence t^{ε}, 1, $t^{-\varepsilon}$ with $\varepsilon \in \{\pm 1\}$.

Theorem 2.1 ([9], page 182) *Let $G^{\#} = \langle G, t;\, t^{-1}at = a^{\varphi}\ (a \in A)\rangle$ be an HNN-extension of G. Then*

(I) *The group G is embedded in $G^{\#}$ by the map $g \mapsto g$. If $g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n = 1$ in $G^{\#}$ where $n \geq 1$, then $g_0,\, t^{\varepsilon_1},\, g_1,\, \ldots,\, t^{\varepsilon_n},\, g_n$ is not reduced.*

(II) *Every $w \in G^{\#}$ has a unique representation as $w = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$ where $g_0,\, t^{\varepsilon_1},\, g_1,\, \ldots,\, t^{\varepsilon_n},\, g_n$ is a normal form.*

The only background needed about basic commutators can be found in [6], Section 11.1.

We will use the following terminology throughout. If X and Y are totally ordered sets, then we can totally order the set $X \times Y$ by

$$(x_1, y_1) < (x_2, y_2) \quad \text{iff} \quad x_1 < x_2 \ \text{ or } \ (x_1 = x_2 \ \&\ y_1 < y_2).$$

This is called the *lexicographic product* of X and Y. If X and Y are right-ordered groups, then so is $X \times Y$ under this ordering; if X and Y are o-groups, then so is $X \times Y$ under this ordering.

3 A first part of construction: the group G_1.

Let $\hat{H}$ be any finitely presented right-orderable group with insoluble word problem; say, $\hat{H} := \langle x_1, \ldots, x_m;\, u_1 = \ldots = u_n = 1\rangle$. (For the existence of such $\hat{H}$, see [2] or [3].) Let $F(\bar{x}) := F(x_1, \ldots, x_m)$ denote the free group on free generators $x_1, \ldots, x_m$ and write $\hat{H} = F(\bar{x})/N$, where N is the normal subgroup of $F(\bar{x})$ generated by $u_1, \ldots, u_n$.

Let G_0 be generated by $x_1, \ldots, x_m,\ b_1, \ldots, b_{2m}$ and have defining relations

$$x_i^{b_j} = x_i^{x_j}, \qquad x_i^{b_{m+j}} = x_i^{x_j^{-1}} \qquad (i, j = 1, \ldots, m). \tag{1}$$

So G_0 is a semidirect product of the free group $F(\bar{x})$ by the free group $F(\bar{b})$ with free generators $b_1, \ldots, b_{2m}$, and N is normalised by $F(\bar{b})$ in G_0.

Let G_1 be generated by $x_1, \ldots, x_m,\ b_1, \ldots, b_{2m}$, and the extra generator t and have defining relations (1) and

$$[t, u_j^g] = 1 \qquad (j = 1, \ldots, n;\ \ g \in F(\bar{x})). \tag{2}$$

So G_1 is an HNN-extension of G_0. By Theorem 2.1, G_0 (and hence $F(\bar{x})$ and $F(\bar{b})$) can be embedded in G_1 in the natural way. We will regard N, $F(\bar{x})$ and $F(\bar{b})$ as subgroups of G_1. By Britton's Lemma,

Lemma 3.1 *If $w \in F(\bar{x})$, then $[t, w] = 1$ in G_1 if and only if $w \in N$.*

Let T_0 be the normal closure of $\langle t\rangle$ in G_1 (so $T_0 = \langle t\rangle^{G_0}$).

For each $\hat{h} \in \hat{H} = F(\bar{x})/N$, choose $h \in F(\bar{x})$ a preimage of $\hat{h}$. We will choose 1 to be the preimage of $\hat{1}$. So $Nh = hN = \hat{h}$ and $H := \{h \in F(\bar{x}) \mid \hat{h} \in \hat{H}\}$ is a transversal for N in $F(\bar{x})$.

For $f_1, f_2 \in F(\bar{x})$ and $v_1(\bar{b}), v_2(\bar{b}) \in F(\bar{b})$, we have

$$t^{v_1(\bar{b})f_1} = t^{v_2(\bar{b})f_2} \quad \text{iff} \quad (v_1(\bar{b}) = v_2(\bar{b})\ \ \&\ \ \hat{f}_1 = \hat{f}_2).$$

Hence, using the normal form for elements of an HNN-extension or Britton's Lemma (see above), one immediately obtains

Lemma 3.2 *T_0 is a free group with free generators t^{vh} $(v \in F(\bar{b});\ h \in H)$.*

4 Preliminaries (Orderability)

Throughout the rest of the paper, we will use $<$ for a two-sided total order and $\prec$ for a right total order on a group.

If G is a right-ordered group, let $G_+ := \{g \in G : g \succ 1\}$, the set of *strictly positive elements of* G. Note that G_+ is a subsemigroup of G and $\{G_+, G_+^{-1}, \{1\}\}$ is a partition of G. If P is a subsemigroup of G and $\{P, P^{-1}, \{1\}\}$ is a partition of G, then G can be right ordered by setting $f \prec g$ if and only if $gf^{-1} \in P$. If G is an o-group, then G_+ is a normal subsemigroup of G with the above properties, and any normal subsemigroup P with these properties can be used to make G an o-group.

Let G be an ordered group and C be a subgroup of G. We say that C is *convex in* G if $g \in C$ whenever there are $c_1, c_2 \in C$ with $c_1 < g < c_2$. The set of convex subgroups of G form a totally ordered set under inclusion ([5], Lemma 3.1.2). For a subgroup H of an ordered group G, we denote by $con(H)$ the *convexification* of H in G; $con(H)$ is the smallest convex subgroup of G which contains H, so $con(H)$ is equal to the intersection of all convex subgroups of G containing H. In two-sided ordered groups, $con(H)$ contains those and only those elements $g \in G$ for which there are $h \in H$ such that $h^{-1} \leq g \leq h$.

We call an o-group *archimedean* if (whenever $g \in G$ and $f^k < g$ for all $k \in \mathbb{Z}$, then $f = 1$). Every archimedean o-group is abelian and is isomorphic to a subgroup of the additive group of real numbers equipped with the usual ordering (Hölder's Theorem; see, *e.g.*, [5], Theorem 4.A).

If G is an o-group and $g, h \in G_+$, we will say that g and h are *archimedean equivalent* if the convex subgroups that they generate are equal; that is, there are $r, s \in \mathbb{Z}_+$ such that $g < h^r$ and $h < g^s$. Thus an o-group G is archimedean if and only if all elements of G_+ are archimedean equivalent.

If G is an o-group and $g, h \in G_+$ with $g^k < h$ for all $k \in \mathbb{Z}_+$, we will write $g \ll h$.

If H is an o-group and the group G acts on H, then we say that the order on H is *G-invariant* if $h > 1$ implies $h^g > 1$ for all $h \in H$ and $g \in G$, and a subgroup K of H is *G-invariant* if $K^g = K$ for all $g \in G$.

We will also need that for any o-group G, the topological completion G^* of G under the order topology is an o-group (see [1]). The o-group G^* can also be realised as the sequential completion of G; that is, every sequence that is a left and right Cauchy sequence is convergent, and all elements of G^* are limits of Cauchy sequences in G (see [7]).

Let F be a free group. The standard way to order F is to use the lower central series $\gamma_k(F)$ ($k \in \mathbb{Z}_+$). The key is that $\gamma_k(F)/\gamma_{k+1}(F)$ is a free abelian group and so can be made into an o-group ($k \in \mathbb{Z}_+$). Since $\bigcap_{k\in\mathbb{Z}_+} \gamma_k(F) = \{1\}$, we can produce a two-sided order on F as follows: Make each $\gamma_k(F)/\gamma_{k+1}(F)$ an o-group. Let $f, g \in F$ with $f \neq g$. Let $s \in \mathbb{Z}_+$ be such that $gf^{-1} \in \gamma_s(F) \setminus \gamma_{s+1}(F)$. Then F is an o-group if we define $f < g$ if and only if $\gamma_{s+1}(F)1 < \gamma_{s+1}(F)gf^{-1}$ in $\gamma_s(F)/\gamma_{s+1}(F)$. We call any order constructed in this way a *standard central order*; so standard central orders on free groups depend only on the orders defined on the set of abelian groups $\{\gamma_k(F)/\gamma_{k+1}(F) \mid k \in \mathbb{Z}_+\}$. For this and further background, also see [4], [5] or [8].

5 Ordering G_1

As noted in the previous section, the free groups $F(\bar{x})$ and $F(\bar{b})$ are two-sided orderable groups. By (1), $F(\bar{b})$ acts by conjugation on $F(\bar{x})$. Hence G_0 is an o-group if we define

$$u(\bar{b})v(\bar{x}) > 1 \quad \text{iff} \quad u(\bar{b}) > 1 \quad \text{or} \quad (u(\bar{b}) = 1 \ \& \ v(\bar{x}) > 1). \tag{3}$$

Thus $G_1/T_0 \cong G_0$ is a two-sided ordered group. Now G_1 is an ordered group with convex subgroup T_0 *if* we can define a G_1-invariant (two-sided) order on T_0 (define $g > 1$ in G_1 if and only if either $gT_0 > T_0$ in G_1/T_0 or $g \geq 1$ in T_0). So it remains to construct a G_1-invariant order on T_0. Note that the construction will not be effective (it doesn't need to be). Really we use only one non-effective step which is confined to the existence of a right order on $\hat{H}$.

To construct such an order on T_0, we first put a *right* order $\prec$ on G_0 as follows. Let $\prec$ be the right total order on $\hat{H}$. Since N is a subgroup of a free group and hence is free, it can be made into an o-group as described in the previous section using any standard central ordering. Define a right order on $F(\bar{x})$ by:

$$f \succ 1 \quad \text{iff} \quad Nf \succ N \quad \text{or} \quad f \in N_+.$$

Next, as described in the previous section, we can put a standard central order on $F(\bar{b})$ so that $b_i < 1$ $(i = 1, \ldots, 2m)$ and the order on the free abelian group $F(\bar{b})/\gamma_2(F(\bar{b}))$ is archimedean. Define

$$v(\bar{b})h \succ 1 \quad \text{iff} \quad v(\bar{b}) \in F(\bar{b})_+ \ \text{ or } \ (v(\bar{b}) = 1 \ \& \ h \in F(\bar{x})_+).$$

This is a well-defined *right* order on the group G_0. Let

$$\Lambda := \{v(\bar{b})h \in G_0 \mid v(\bar{b}) \in F(\bar{b}), \ h \in H\}$$

with the inherited order.

By Lemma 3.2, $\{t^g \mid g \in \Lambda\}$ is a set of free generators for T_0; it inherits a total order $\ll$ given by:

$$t^f \ll t^g \quad \text{iff} \quad f \prec g \quad (f, g \in \Lambda).$$

So for each $a \in G_0$, $t^{fa} \ll t^{ga}$ if and only if $t^f \ll t^g$.

For each $g \in \Lambda$, consider the normal subgroups of T_0

$$C_g := \langle t^f \mid f \in \Lambda, \ \ f \prec g\rangle^{T_0} \quad \text{and} \quad C(g) := \langle t^f \mid f \in \Lambda, \ \ f \preceq g\rangle^{T_0}.$$

We will use this set of normal subgroups of T_0 to define a two-sided order on T_0 that is G_1-invariant (see Lemma 5.4).

For each $g \in \Lambda$, let

$$K_g := \langle t^f \mid f \in \Lambda,\ \ f \succeq g\rangle \quad \text{and} \quad K(g) := \langle t^f \mid f \in \Lambda,\ \ f \succ g\rangle.$$

By Lemma 3.2, K_g and $K(g)$ are free groups on the indicated generators ($g \in \Lambda$).

Lemma 5.1 *Let $g \in \Lambda$.*

(i) $[t^f, t^{f'}] \in C(g)$ *for each $f, f' \in G_0$ with $f \preceq g$.*

(ii) $T_0/C(g) \cong K(g)$, $T_0/C_g \cong K_g$; *and T_0 is isomorphic to the semidirect product $C(g) \rtimes K(g)$ as well as to the semidirect product $C_g \rtimes K_g$.*

(iii) $C(g)^f = C(gf)$ *and* $C_g^f = C_{gf}$ *for each $f \in G_0$.*

Moreover,

(iv) $T_0 = \bigcup_{g\in\Lambda} C(g) = \bigcup_{g\in\Lambda} C_g$, *and*

(v) $\bigcap_{g\in\Lambda} C(g) = \bigcap_{g\in\Lambda} C_g = \{1\}$.

Proof: This is immediate by the definitions and our choice of ordering of the generators of T_0. //

Lemma 5.2 *For any $w \in T_0 \setminus \{1\}$, there is a unique $g \in \Lambda$ such that $w \in C(g) \setminus C_g$*

Proof: Any $w \in T_0 \setminus \{1\}$ can be written as a reduced word in some $t^{f_1}, \ldots, t^{f_n}$ with $f_1 \prec \ldots \prec f_n$ in Λ. Clearly, $w \in C(f_n)$. Choose the minimal $k \in \{1, \ldots, n\}$ such that $w \in C(f_k)$. Then $w \notin C_{f_k}$ (otherwise, $w \in \langle t^f \mid f \in \Lambda,\ \ f \prec f_k\rangle^{T_0}$; so $w \in C(f_{k-1})$, a contradiction). //

We now show how to construct a G_1-invariant order on T_0. We first define a lexicographic order on the abelian groups $\gamma_k(T_0)/\gamma_{k+1}(T_0)$ ($k \in \mathbb{Z}_+$). For this we associate with any basic commutator $w(t^{f_1}, t^{f_2}, \ldots, t^{f_s})$, the monomial $s(w) = f_1^{n_1} f_2^{n_2} \ldots f_s^{n_s}$, where n_i is the number of occurrences of t^{f_i} in w and $f_1 \succ f_2 \succ \ldots \succ f_s$. For example, if $w = [t^{f_1}, [t^{f_1}, t^{f_5}], [t^{f_2}, t^{f_3}, t^{f_5}, t^{f_5}]]$ and $f_1 \succ \ldots \succ f_5$, then $s(w) = f_1^2 f_2 f_3 f_5^3$. We put the lexicographical order on the set of such monomials. For basic commutators $w_1, w_2 \in \gamma_k(T_0) \setminus \gamma_{k+1}(T_0)$, define

$$w_1 \ll w_2 \quad \text{if} \quad s(w_1) < s(w_2);$$

and fix $w_1 \ll w_2$ arbitrarily if $s(w_1) = s(w_2)$. Now let $\leq_0$ be the standard central order on T_0 built from the total orders on $\gamma_k(T_0)/\gamma_{k+1}(T_0)$ ($k \in \mathbb{Z}_+$) described above. The order $\leq_0$ induces a total order on the subgroup K_1. The isomorphism given in Lemma 5.1(ii) induces a total order on the group T_0/C_1. This, in turn, induces an order on its subgroup $C(1)/C_1$. We also denote this order by $\leq_0$. Let w be a non-trivial element from T_0. By Lemma 5.2, there is a unique $g \in \Lambda$ such that $w \in C(g) \setminus C_g$. Define

$$1 <_1 w \quad \text{iff} \quad C_1 <_0 C_1 w^{g^{-1}} \text{ in } C(1)/C_1. \tag{4}$$

Lemma 5.3 *Subgroups C_g and $C(g)$ are convex in G_1 under order $\leq_1$ for all $g \in \Lambda$.*

Proof: Since T_0 is convex in G_1, it is sufficient to prove that C_g and $C(g)$ are convex in T_0. Let $1 <_1 u <_1 h$, $h \in C(g)$. This gives $1 <_1 hu^{-1}$. By Lemma 5.2, we find $f \in \Lambda$ such that $u \in G(f) \setminus G_f$. If $u \notin C(g)$, then $g \prec f$ and so $C(g) \leq C_f$. Thus $u^{-1} \in G(f) \setminus G_f$ and $hu^{-1} \in G(f) \setminus G_f$. By (4), $C_1 <_0 C_1(hu^{-1})^{f^{-1}} = C_1 h^{f^{-1}} u^{-f^{-1}} = C_1 u^{-f^{-1}}$ and so $1 <_1 u^{-1}$, a contradiction. Therefore $u \in C(g)$ and $C(g)$ is convex. The convexity of C_g follows and is left to the reader.//

Lemma 5.4 *The order $\leq_1$ defined in (4) is a G_1-invariant two-sided order on T_0.*

Proof: Assume that $1 <_1 w_1, w_2$. By Lemma 5.2, there are $g_1, g_2 \in \Lambda$ such that $w_1 \in C(g_1) \setminus C_{g_1}$ and $w_2 \in C(g_2) \setminus C_{g_2}$. If $g_1 \prec g_2$, then $w_1 w_2 \in C(g_2) \setminus C_{g_2}$ and $C_1 <_0 C_1 w_2^{g_2^{-1}} = C_1(w_1 w_2)^{g_2^{-1}}$. Hence $1 <_1 w_1 w_2$. Similarly, $1 <_1 w_1 w_2$ if $g_2 \prec g_1$. If $g_1 = g_2$, then $C_1 <_0 C_1 w_1^{g_2^{-1}}$ and $C_1 <_0 C_1 w_2^{g_2^{-1}}$; so $C_1 <_0 C_1(w_1 w_2)^{g_2^{-1}}$, whence $1 <_1 w_1 w_2$.

Let $w \in T_0$ and $1 <_1 w$. By Lemma 5.2, there is $g \in \Lambda$ such that $w \in C(g) \setminus C_g$ and $C_1 <_0 C_1 w^{g^{-1}}$. Since C_g is a normal subgroup of T_0, we get $w^v \in C(g) \setminus C_g$ for any $v \in T_0$. To prove that $<_1$ is a two-sided total order on T_0, we must show that $C_1 <_0 C_1(w^v)^{g^{-1}}$. This immediately follows from the definition of a standard central order on T_0 because $\gamma_{k+1}(T_0)w = \gamma_{k+1}(T_0)w^v$ provided that $w \in \gamma_k(T_0) \setminus \gamma_{k+1}(T_0)$. Thus the order $\leq_1$ is T_0-invariant; *i.e.*, $\leq_1$ is a two-sided order on T_0.

Let $f \in \Lambda$. Now $w^f \in C(gf) \setminus C_{gf}$ and $(w^f)^{(gf)^{-1}} = w^{g^{-1}}$. Hence $1 <_1 w^f$. Thus the order $\leq_1$ is G_1-invariant. //

6 The second part of construction: the group G.

Denote by T_0^* the topological (sequential) completion of T_0 under the interval topology induced by the order on T_0 defined in Section 5. We regard T_0^* as an o-group with the order $\leq_*$ extending the initial order $\leq_1$ of the group T_0. Since convex subgroups $C(g)$ are normal in T_0 (by Lemma 5.3) and their intersection is trivial (by Lemma 5.1 (v)), the right and the left topological spaces coincide on T_0. In this case, any right Cauchy sequence is a left Cauchy sequence and *vice versa* and we can simply write "Cauchy sequence" without ambiguity. Conjugation by $g \in G_1$ preserves the order on T_0, so maps open intervals on open intervals; hence conjugation is a continuous operation. This allows us to define an order-preserving action by $G_0 < G_1$ on T_0^* coinciding with conjugation on T_0. Let $\{c_k \mid k \in \mathbb{Z}_+\}$ be a Cauchy sequence in T_0. Then

$$(\lim_{k\to\infty} c_k)^g = \lim_{k\to\infty} c_k^g \quad (\lim_{k\to\infty} c_k \in T_0^*,\ c_k \in T_0,\ g \in G_0). \tag{5}$$

We define now $2m$ order-preserving automorphism $y_1, \dots, y_{2m}$ on T_0^*. First, let

$$(t^g)^{y_i} = [b_i, t]^g \quad (g \in \Lambda;\ i = 1, \dots, 2m). \tag{6}$$

By Lemma 3.2, the set $\{t^g \mid g \in \Lambda\}$ freely generates T_0 so each y_i uniquely defines an endomorphism $y_i : T_0 \to T_0$,

$$w(t^{g_1}, \dots, t^{g_k})^{y_i} = w(t^{g_1 y_i}, \dots, t^{g_k y_i}) \quad (w(t^{g_1}, \dots, t^{g_k}) \in T_0,\ i = 1, \dots, 2m).$$

It follows from (6) that

$$w^{y_i g} = w^{g y_i} \quad \text{for all } w \in T_0,\ g \in \Lambda;\ i = 1, \dots, 2m. \tag{7}$$

Lemma 6.1 *The endomorphisms $y_1, \dots, y_{2m}$ of T_0 preserve the order $\leq_1$ on T_0; moreover w^{y_i} is archimedean equivalent to w for all $w \in T_0$, $i = 1, \dots, 2m$.*

Proof: Consider a basic commutator $d \in \gamma_k(T_0)/\gamma_{k+1}(T_0)$, $k \in \mathbb{Z}_+$ and assume $1 <_0 d$. Applying (6) to $d = [t^{f_1}, \dots, [\dots, \dots], t^{f_s}]$, we obtain (modulo $\gamma_{k+1}(T_0)$) that

$$d^{y_i} = [t^{-b_i f_1} t^{f_1}, \dots, [\dots, \dots], t^{-b_i f_s} t^{f_s}] = d \cdot d_1 \dots d_p, \tag{8}$$

where $s(d_1), \dots, s(d_p) < s(d)$. By the definition of order $\leq_0$, all $d_1, \dots, d_p \ll d$ and so $d^{y_i} >_0 1$ and d^{y_i} is archimedean equivalent to d ($i = 1, \dots, 2m$). Now

we repeat arguments from the proof of Lemma 5.4. Let $k \in \mathbb{Z}_+$ be such that $w^{g^{-1}} \in \gamma_k(T_0) \setminus \gamma_{k+1}(T_0)$. So there are basic commutators $c_1 \gg \ldots \gg c_\ell$ in $\gamma_k(T_0)$ and $r_1, \ldots, r_\ell \in \mathbb{Z} \setminus \{0\}$ such that

$$w^{g^{-1}} = c_1^{r_1} \ldots c_\ell^{r_\ell} w',$$

with $w' \in \gamma_{k+1}(T_0)$. Since $1 <_1 w$, we have $r_1 \in \mathbb{Z}_+$ by the definition of the order $\leq_1$. Congugating $w^{g^{-1}}$ by y_i and using (7) gives $c_1^{r_1 y_i} >_1 1$ and $c_1^{y_i}$ is archimedean equivalent to c_1 by (8). Thus the order $\leq_1$ is invariant under each of the endomorphisms $y_1, \ldots, y_{2m}$ and w^{y_i} is archimedean equivalent to w for all $w \in T_0$. //

Lemma 6.2 *The endomorphisms $y_1, \ldots, y_{2m}$ of T_0 extend to order-preserving automorphisms of T_0^*.*

Proof: First we show that endomorphism y_i maps a Cauchy sequence on a Cauchy sequence. Indeed, if $\{c_k\}_{k \in \mathbb{Z}_+}$ is a Cauchy sequence, then for any $g \in \Lambda$ there is $k_g \in \mathbb{Z}_+$ such that $c_{k_g} c_{k_g+s}^{-1} \in C(g)$ for all $s \in \mathbb{Z}_+$. By Lemma 6.1, c_k is archimedean equivalent to $c_k^{y_i}$ and so $c_{k_g}^{y_i} c_{k_g+s}^{-y_i} \in C(g)$. Thus $\{c_k^{y_i}\}_{k \in \mathbb{Z}_+}$ is a Cauchy sequence. For each $g \in G_0$ and $i = 1, \ldots, 2m$ consider sequences $\{c_k(g,i)\}_{k \in \mathbb{Z}_+}$ defined by

$$c_0(g,i) := t^g, \quad c_k(g,i) := t^{b_i^k g} c_{k-1}(g,i), \qquad k \in \mathbb{Z}_+. \tag{9}$$

The ordering on G_1 ensures that $\{c_k(g,i)\}_{k \in \mathbb{Z}_+}$ is a Cauchy sequence in T_0 and so $\tilde{c}(g,i) := \lim_{k \to \infty} c_k(g,i) \in T_0^*$ for every $g \in G_0$ and $i = 1, \ldots, 2m$. By routine verification,

$$(c_k(g,i))^{y_i} = t^{-b_i^{k+1} g} t^g \quad (k \in \mathbb{Z}_+).$$

By the sequential completeness of T_0^* we have

$$(\tilde{c}(g,i))^{y_i} = \lim_{k \to \infty} (c_k(g,i))^{y_i} = t^g.$$

It follows now that each element $w = w(t^{g_1}, \ldots, t^{g_k}) \in T_0$ has a unique preimage $w^{y_i^{-1}} = w(t^{g_1 y_i^{-1}}, \ldots, t^{g_k y_i^{-1}}) \in T_0^*$ $(i = 1, \ldots, 2m)$. Given a Cauchy sequence $\{w_k\}_{k \in \mathbb{Z}_+}$, the sequence $\{w_k^{y_i^{-1}}\}_{k \in \mathbb{Z}_+}$ is also a Cauchy sequence. Thus $(\lim_{k \to \infty} w_k)^{y_i^{-1}} = \lim_{k \to \infty} w_k^{y_i^{-1}} \in T_0^*$. Hence each $\tilde{w} \in T_0^*$ has a unique preimage $\tilde{w}^{y_i^{-1}}$. So y_i extends to an order-preserving automorphism of T_0^* $(i = 1, \ldots, 2m)$. //

Let $F(\bar{y})$ be a free group on free generators $y_1, \ldots, y_{2m}$. By Lemma 6.2, $F(\bar{y})$ acts on T_0^* by order-preserving automorphisms whose action on T_0 is given by (6). We also defined the order-preserving action of G_0 on T_0^* above in (5). Since these two actions are continuous and commute on T_0, they commute on T_0^*, and we can form the semidirect product $T_0^* \rtimes (G_0 \times F(\bar{y}))$. The group $T_0^* \rtimes (G_0 \times F(\bar{y}))$ is orderable because T_0^* and $G_0 \times F(\bar{y})$ are orderable and $G_0 \times F(\bar{y})$ acts an T_0^* by order-preserving automorphisms (with respect to $\leq_*$). Define

$$G := \langle t, x_1, \ldots, x_m, b_1, \ldots, b_{2m}, y_1, \ldots, y_{2m} \rangle < T_0^* \rtimes (G_0 \times F(\bar{y})), \tag{10}$$

and

$$T := \langle t \rangle^G < T_0^*. \tag{11}$$

Thus we obtain

Proposition 6.3 *The group G is orderable.*

It follows from the defintion (10) the the group G is generated by $5m+1$ elements: $t, x_1, \ldots, x_m, b_1, \ldots, b_{2m}, y_1, \ldots, y_{2m}$ and G contains the subgroup $\langle t, x_1, \ldots, x_m, b_1, \ldots, b_{2m} \rangle$ isomorphic with G_1. Hence G satisfies relations (1) and (2) which hold in G_1. By the construction, G satisfies relations (6) and (7). In addition, G satisfies the relations

$$[x_i, y_j] = 1 \qquad (i = 1, \ldots, m; \;\; j = 1, \ldots, 2m), \tag{12}$$

and

$$[b_i, y_j] = 1 \qquad (i, j = 1, \ldots, 2m) \tag{13}$$

which follow from (10). Finally we have that the group G satisfies relations (1), (2), (6), (7), (12), and (13). We extract a finite subset from this set of relations, namely the relation (1), (12), (13) and relations

$$t^{y_i} = [b_i, t] \qquad (i = 1, \ldots, 2m), \tag{14}$$

and

$$[t, u_j] = 1 \qquad (j = 1, \ldots, n). \tag{15}$$

Lemma 6.4 *The two sets of relations relations (1), (2), (6), (7), (12), (13) and (1), (12) – (15) are equivalent.*

Proof: First we show that relations (1), (2), (6), (7), (12), (13) follow from relations (1) and (12) — (15).

Conjugating $[t, u_j] = 1$ by y_i and then by b_i^{-1} (and by y_{i+m} and then by b_{i+m}^{-1}) and using (1), (12) — (14), we obtain $[t, u_j^{x_i^{\pm 1}}] = 1$ ($j = 1, \ldots, n$; $i = 1, \ldots, m$). An easy induction now gives that $[t, u_j^{w(\bar{x})}] = 1$ for all $j \in \{1, \ldots, n\}$ and $w(\bar{x}) \in F(\bar{x})$. Hence the relations (2) hold in G. Relations (6) follow from (14) and (12), (13). Relations (7) follow from (12), (13).

Now, relations (14), (15) are the partial cases of relations (6) and (2) respectively. The lemma follows. //

Note that $G \cong T \rtimes (G_0 \times F(\bar{y}))$ by construction and so

$$G/T \cong G_0 \times F(\bar{y}) \tag{16}$$

However the isomorphsm (16) follows immediately from (1), (12) – (15)

Let $w \in F(\bar{x})$. Since $[t, w] = 1$ in G if and only if $w \in N$ (by Lemma 3.1), it follows that

Proposition 6.5 *The group G has insoluble word problem.*

Therefore, the rest of the paper is devoted to showing that G is finitely presented. By Lemma 6.4, it is enough to prove that relations (1), (2), (6), (7), (12), (13) completely define the group G. To achieve this, in the next section we construct a generating set for the subgroup T. We complete this section by considering convexifications (convex closures) of the subgroups C_g and $C(g)$ in the group G. For $g \in \Lambda$, denote the convexifications of C_g and $C(g)$ in G by D_g and $D(g)$, respectively. So

$$D_g = con(C_g) \text{ and } D(g) = con(C(g)).$$

Lemma 6.6 *For each $g \in \Lambda$,*

(i) *for each $u \in T$ there exists $v \in T_0$ such that $D_g u = D_g v$ and $D(g)u = D(g)v$;*

(ii) *D_g and $D(g)$ are $\langle T, F(\bar{y})\rangle$-invariant; ;*

(iii) *$D(g) \bigcap G_1 = C(g)$ and $D_g \bigcap G_1 = C_g$;*

(iv) *$T/D_g \cong K_g$ and $T/D(g) \cong K(g)$;*

(v) *$T \cong D_g \rtimes K_g \cong D(g) \rtimes K(g)$.*

(vi) $D_g^f = D_{gf}$ *and* $D(g)^f = D(gf)$ *for all* $f \in G_0$.

Moreover,

(vii) $\bigcup_{g\in\Lambda} D(g) = T$.

Proof: Let $u \in T$, say $u = \lim_{k\to\infty} v_k$ for some Cauchy sequence $\{v_k\}_{k\in\mathbb{Z}_+}$ in T_0. Then there exists $k = k(g) \in \mathbb{Z}_+$ such that $u \cdot v_{k(g)}^{-1} \in C_g$. Since $C_g \leq D_g$, we get $D_g u = D_g v_{k(g)}$ and (i) is proved for D_g.

Now $D_g = con(C_g)$ is normalised by T_0 since C_g is normal in T_0. For $u \in T$, by (i) there is $v \in T_0$ such that $uv^{-1} \in D_g$. Thus $D_g^u = D_g^v = D_g$ and D_g is normal in T. Since w_i^y is archimedean equivalent to w and D_g is convex, it follows that D_g is $F(\bar{y})$-invariant. Hence (ii) holds for D_g.

Let $u \in G_1$ with $h_1 \leq u \leq h_2$ for some $h_1, h_2 \in C_g$. Since C_g is convex in G_1, we get $u \in C_g$. This gives (iii) for D_g.

Applying (i), we have $T = D_g T = D_g T_0$. By (iii), $D_g \bigcap T_0 = C_g$ and hence $T/D_g = D_g T_0/D_g \cong T_0/(D_g \bigcap T_0) = T_0/C_g$.

By Lemma 5.1, $T_0/C_g \cong K_g$; so $T/D_g \cong K_g$. Since $D_g \bigcap K_g \leq T_0$, we have $D_g \bigcap T_0 = C_g$. Moreover, $D_g \bigcap K_g = \{1\}$ since $C_g \bigcap K_g = \{1\}$. This gives (v) for D_g.

By Lemma 5.1 (iii), we have

$$D_g^f = con(C_g)^f = con(C_g^f) = con(C_{gf}) = D_{gf}.$$

By Lemma 5.1 (iv), $T_0 = \bigcup_{g\in\Lambda} C_g \subset \bigcup_{g\in\Lambda} D_g$. Since $\bigcup_{g\in\Lambda} D_g$ is convex and each $w \in T$ is archimedean equivalent to some $u \in T_0$, we get $T \subseteq \bigcup_{g\in\Lambda} D_g$. This gives (vii).

Similarly, we obtain (i) – (vii) for $D(g)$. Hence the lemma follows.//

7 A generating set for T.

We will need two identities that follow immediately from (13) and (14): for $i = 1, \ldots, 2m$,

$$t^{y_i^{-1} b_i} = t^{y_i^{-1}} t^{-1}, \tag{17}$$

and

$$t^{y_i^{-1} b_i^{-1}} = t^{y_i^{-1}} t^{b_i^{-1}}. \tag{18}$$

Lemma 7.1 *T has generators*

$$t^{\alpha(\bar{y})v(\bar{b})h(\bar{x})}, \tag{19}$$

where $h(\bar{x}) \in H$, $v(\bar{b}) \in F(\bar{b})$ and $\alpha(\bar{y})$ is either empty or, for some $i \in \{1, \ldots, 2m\}$, $\alpha(\bar{y})$ is a non-trivial element of $F(\bar{y})$ that begins with y_i^{-1} and $v(\bar{b})$ does not begin with $b_i^{\pm 1}$.

Proof: The elements of T of the form

$$t^{\alpha(\bar{y})v(\bar{b})h(\bar{x})} \quad (\alpha(\bar{y}) \in F(\bar{y}),\, v(\bar{b}) \in F(\bar{b}),\, h(\bar{x}) \in F(\bar{x})) \tag{20}$$

generate T. Since $t^{\alpha(\bar{y})v(\bar{b})f_1(\bar{x})} = t^{\alpha(\bar{y})v(\bar{b})f_2(\bar{x})}$ if $Nf_1 = Nf_2$, we may assume that $h(\bar{x}) \in H$ in (20).

We prove by induction on pairs of natural numbers $(\ell(\alpha), \ell(v))$ (ordered lexicographically) that $t^{\alpha(\bar{y})v(\bar{b})h}$ can be written as a product of conjugates of t and t^{-1} all of the form described in (19).

If $\ell(\alpha) = 0$, it already has the desired form. If $\alpha(\bar{y})$ begins with y_i for some $i \in \{1, \ldots, 2m\}$, $\alpha := y_i \cdot \alpha'$ then

$$t^{\alpha(\bar{y})v(\bar{b})h} = t^{y_i\alpha'(\bar{y})v(\bar{b})h}.$$

By (14) and (13),

$$t^{y_i\alpha'(\bar{y})v(\bar{b})h} = t^{-b_i\alpha'(\bar{y})v(\bar{b})h} \cdot t^{\alpha'(\bar{y})v(\bar{b})h} = t^{-\alpha'(\bar{y})b_iv(\bar{b})h} \cdot t^{\alpha'(\bar{y})v(\bar{b})h}.$$

The two conjugators on the right-hand side have $\bar{y}$-length $\ell(\alpha') < \ell(\alpha)$, so (by induction) each can be written as a product of the desired form. Hence, so can $t^{\alpha(\bar{y})v(\bar{b})h}$.

We may therefore assume that $\alpha(\bar{y})$ begins with y_i^{-1} for some $i \in \{1, \ldots, 2m\}$; say, $\alpha(\bar{y}) := y_i^{-1}\alpha'(\bar{y})$. If $v(\bar{b})$ does not begin with b_i or b_i^{-1}, then $t^{\alpha(\bar{y})v(\bar{b})h}$ has the desired form.

If, on the other hand, $v = b_iv'$ with $\ell(v') < \ell(v)$, then by (13) and (17)

$$t^{\alpha(\bar{y})v(\bar{b})h} = t^{y_i^{-1}b_i\alpha'(\bar{y})v'(\bar{b})h} = t^{\alpha(\bar{y})v'(\bar{b})h} \cdot t^{-\alpha'(\bar{y})v'(\bar{b})h}$$

and $(\ell(\alpha'), \ell(v')) < (\ell(\alpha), \ell(v')) < (\ell(\alpha), \ell(v))$. By induction, we get that $t^{\alpha(\bar{y})v(\bar{b})h}$ is the product of of conjugates of t and t^{-1} all of the form given in (19). If v begins with b_i^{-1}, we can repeat the above argument using (18) instead of (17). This completes the proof that the elements displayed in (19) are indeed generators of T. //

We will write *Gen* for the set of generators described in (19); in Corollary 7.5, we will prove that they form a *free* generating set.

We will frequently use that for each $f, g \in \Lambda$ and $i \in \{1, \ldots, 2m\}$, there is $k = k_{i,f,g} \in \mathbb{Z}_+$ such that $b_i^k f \prec g$. This is immediate from the definition of $\prec$.

For each $g \in \Lambda$, let $A(g) := (T/D(g))/\gamma_2(T/D(g))$, a free abelian group by Lemma 6.6 (iv).

Remark 7.2 Let $g \prec b_i^k$ and $a_{k,p}$ be the coefficient of b_i^k in $t^{y_i^{-p}} D(g)$ in $A(g)$. Then $a_{k,1} = 1$ and $a_{k,p+1} = \sum_{\ell \leq k} a_{\ell,p}$. If $k \in \mathbb{N}$, $r \geq 2$ and $p_1, \ldots, p_r \in \mathbb{Z}_+$ are distinct with $g \prec b_i^{k+r}$, then this recursive formula immediately gives that the matrix $(a_{\ell,p})$ with $1 \leq \ell, p \leq P$ has determinant 1 where $P := \max\{p_s \mid s = 1, \ldots, r\}$. Thus the r columns (a_{ℓ,p_s}) are linearly independent, whence, using the recursive formula again, the $r \times r$ matrix (a_{k+j,p_s}) has non-zero determinant.

Lemma 7.3 *Let $\alpha_1, \ldots, \alpha_r \in F(\bar{y})$ be distinct with each α_s either empty or beginning with some $y_{i_s}^{-1}$. Then there is $g \in \Lambda$ such that*

$$\{t^{\alpha_1} D(g), \ldots, t^{\alpha_r} D(g)\} \quad \textit{is linearly independent in} \ \ A(g).$$

Proof: Induction on r.

Let $r = 1$. By Lemma 6.6(I), if $g \prec 1$ and $g \succ b_1, \ldots, b_{2m}$, then $t^{\alpha_1} D(g) = tD(g)$. The result follows at once in this case.

Assume the result if $r < r_0$ and let $t^{\alpha_1}, \ldots, t^{\alpha_{r_0}}$ satisfy the hypotheses of the Lemma.

If some α_{s_0} is empty, let $P_0 = \{t^{\alpha_{s_0}}\} = \{t\}$. For all other values of s, write $\alpha_s = y_{i_s}^{-1} \beta_s$ where β_s does not begin with y_{i_s}. Let $P_i = \{t^{\alpha_s} \mid i_s = i\}$ $(i = 1, \ldots, 2m)$. The non-empty sets among $P_0, \ldots, P_{2m}$ partition $t^{\alpha_1}, \ldots, t^{\alpha_{r_0}}$. Now, in $A(g)$, we have that $t^{(\alpha_s)_{1,g}}$ is a product which has terms ending in b_{i_s}. By Lemma 3.2, the set of elements in P_i is linearly independent in $A(g)$ of the set of elements in all the remaining $P_{i'}$ $(i' \neq i)$. So if there are distinct i, j with P_i and P_j non-empty, the original set is linearly independent in $A(g)$ by the induction hypothesis. We may therefore assume that there is a unique $i \in \{1, \ldots, 2m\}$ such that $P_i \neq \emptyset$.

Let $\alpha_s(\bar{y})$ begin with $y_i^{-n_{1,s}}$ $(n_{1,s} \in \mathbb{Z}_+)$ for $s = 1, \ldots, r_0$; say

$$\alpha_s(\bar{y}) = y_i^{-n_{1,s}} \ldots \ y_{i_{q_s,s}}^{n_{q_s,s}}.$$

For each $n \in \mathbb{Z}_+$, let $Q_n := \{s \in \{1, \dots, r_0\} \mid n_{1,s} = n\}$. By Remark 7.2, the set of elements t^{α_s} with $s \in Q_n$ is linearly independent of $\{t^{\alpha_{s'}} \mid s' \notin Q_n\}$. So the lemma follows by induction unless $n_{1,s} = n_{1,1}$ for all $s = 1, \dots, r_0$. We therefore assume that

$$\alpha_s(\bar{y}) = y_i^{-n_{1,1}} \dots y_{i_{q_s,s}}^{n_{q_s,s}} =: y_i^{-n_{1,1}} \delta_s,$$

where δ_s starts with $y_{j_s}^{\pm 1}$ with $j_s \neq i$ $(s = 1, \dots, r_0)$.

We prove the result for r_0 by induction on $\ell = \max\{\ell(\alpha_1), \dots, \ell(\alpha_{r_0})\}$. Now if $g \prec b_i^{n_{1,1}}$, in $A(g)$ we have a linear combination of elements all of the form $t^{\delta_1 b_i^k}, \dots, t^{\delta_s b_i^k}$. Moreover, the sets $\{t^{(\delta_1)_{b_i^k,g} b_i^k}, \dots, t^{(\delta_s)_{b_i^k,g} b_i^k}\}$ for distinct values of k are linearly independent by Lemma 3.2. Hence if $N \in \mathbb{Z}_+$ is sufficiently large and $g \prec b_1^N, \dots, b_{2m}^N$, we obtain that $\{t^{\alpha_1} D(g), \dots, t^{\alpha_{r_0}} D(g)\}$ is linearly independent in $A(g)$ if and only if $\{t^{\delta_1} D(g), \dots, t^{\delta_{r_0}} D(g)\}$ is. But $\max\{\ell(\delta_1), \dots, \ell(\delta_{r_0})\} < \ell$, so the lemma follows by induction. //

Corollary 7.4 *Let $t^{\alpha_1 f_1}, \dots, t^{\alpha_r f_r} \in Gen$ be distinct with $f_1, \dots, f_r \in G_0$ and $\alpha_1, \dots, \alpha_r \in F(\bar{y})$. Then there is $g \in \Lambda$ such that*

$$\{t^{\alpha_1 f_1} D(g), \dots, t^{\alpha_r f_r} D(g)\} \quad \textit{is linearly independent in } A(g).$$

Proof: Let $f_s = v_s(\bar{b}) h_s$ and any non-empty $\alpha_s =: y_{i_s}^{-1} \beta_s$ $(s = 1, \dots, r)$. Let $N \in \mathbb{Z}_+$ be sufficiently large with $N > 2(\ell(v_1) + \ldots + \ell(v_r))$. If $g \prec b_1^{2N}, \dots, b_{2m}^{2N}$ and $s \in \{1, \dots, r\}$ with α_s non-empty, then $t^{\alpha_s v_s h_s} D(g) = t^{(\alpha_s)_{f_s,g} v_s h_s} D(g)$ contains a term with the conjugator ending $b_{i_s}^N v_s h_s$; this portion, in reduced form, begins with $b_{i_s}^{k_s}$ for some $k_s \geq N/2 > \ell(v_s)$. By Lemma 3.2 and the choice of N, we deduce that all h_s are equal as are all v_s; moreover, $\{t^{\alpha_1 f_1} D(g), \dots, t^{\alpha_r f_r} D(g)\}$ is a linearly independent set in $A(g)$ if and only if $\{t^{\alpha_1} D(g), \dots, t^{\alpha_r} D(g)\}$ is. The corollary now follows from Lemma 7.3. //

We now use Nielsen's method (see, *e.g.*, [9], Chapter 1 Section 2) to lift Corollary 7.4 to the non-abelian case.

Corollary 7.5 *Distinct elements $t^{\alpha_1 f_1}, \dots, t^{\alpha_r f_r} \in Gen$ generate a free subgroup.*

Proof: It is enough to prove that there is $g \in \Lambda$ such that $u_1 = t^{\alpha_1 f_1} D(g)$, …, $u_r = t^{\alpha_r f_r} D(g)$ are free generators of the free subgroup $\langle u_1, \dots, u_r \rangle$ of $T/D(g)$. Indeed, by Corollary 7.4, there is $g \in \Lambda$ with $\{u_1, \dots, u_r\}$ linearly independent in $A(g)$. Let U be the subgroup of T generated by $t^{\alpha_1 f_1}, \dots, t^{\alpha_r f_r}$.

By Lemma 6.6 (iv), $D(g)U/D(g)$ is a subgroup of the free group $T/D(g)$ of rank *at most* r. But $u_1, \dots, u_r$ are linearly independent in $A(g)$ and so in $U/\gamma_2(U)$. So the free group $D(g)U/D(g)$ has rank r. Now $u_1, \dots, u_r$ generate the subgroup $D(g)U/D(g)$ of the free group of rank r in $T/D(g)$. By [9], Chapter 1, Proposition 2.7, we get that $\{u_1, \dots, u_r\}$ is a free generating set for $D(g)U/D(g)$. //

Proposition 7.6 *The group G is fintely presented.*

Proof: We claim that G has presentation (1), (12) – (15) in generators t, $x_1, \dots, x_m$, $b_1, \dots, b_m$, $y_1, \dots, y_{2m}$. By Lemma 6.4, we can also use relations (2), (6), and (7) that follow from (1), (12) – (15). Assume $w(t, \bar{x}, \bar{b}, \bar{y}) = 1$ in G. Collecting x_i, b_j, and y_k to the left and using relations (1), (12) and (13), we can rewrite w in the form $w = w_1(\bar{x})w_2(\bar{b})w_3(\bar{y})u$ where $u \in T$ and w_1, w_2, w_3 are reduced words in indicated generators. By (16), the words w_1, w_2, w_3 are empty. Thus $w \in T$.

Using Lemma 7.1, we write w in generators $t^{\alpha_1 f_1}, \dots, t^{\alpha_k f_k} \in Gen$. By Corrolary 7.5, the subgroup $\langle t^{\alpha_1 f_1}, \dots, t^{\alpha_k f_k} \rangle$ is free; so w is the empty word. //

Theorem A now follows from Propositions 6.3, 6.5, 7.6.

We observe that the above proof actually gives

Theorem B *Let $\hat{H}$ be a right-orderable finitely presented group on m generators and ε be the natural homomorphism from the free group F_m onto $\hat{H}$. Then the semidirect product $\langle t^{\hat{h}} \mid \hat{h} \in \hat{H} \rangle \rtimes F$ defined by automorphisms $(t^{\hat{h}})^f = t^{\hat{h}\varepsilon(f)}$ is embeddable in a (two-sided) orderable finitely presented group.*

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